The geometrical form for the string space-time action *

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Abstract

In the present article, we derive the space-time action of the bosonic string in terms of geometrical quantities. First, we study the space-time geometry felt by probe bosonic string moving in antisymmetric and dilaton background fields. We show that the presence of the antisymmetric field leads to the space-time torsion, and the presence of the dilaton field leads to the space-time nonmetricity. Using these results we obtain the integration measure for space-time with stringy nonmetricity, requiring its preservation under parallel transport. We derive the Lagrangian depending on stringy curvature, torsion and nonmetricity.

1 Introduction

The general relativity is described in terms of torsion free and metric compatible connection. There are many generalization of this theory which include nontrivial contribution of torsion and nonmetricity [1].

We are interesting in the theory of gravity obtained from string theory, describing the massless states of the closed bosonic string. Beside the metric tensor $G_{\mu\nu}$, it contains the antisymmetric tensor $B_{\mu\nu}$ and the dilaton field Φ. The space-time field equations of this theory can be derived from the requirement of Weyl invariance of the quantum world-sheet theory, as a condition of consistent string theory [2]. It is nontrivial fact that these field equations can be obtained from a single space-time action. Consequently, the quantum conformal invariance of the world-sheet leads to the generalized space-time Einstein equations and corresponding action. The question is whether there exists the geometrical interpretation of this action? In our interpretation it means the existence of generalized connection, so that the above action can be written in terms of corresponding generalized curvature, torsion and nonmetricity.

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There was many attempts to achieve this goal, expressing this action in terms of geometrical quantities. In ref.[3] there is a restriction on the space-time dimensions (D=2 and D=4), in order to use Hodge dual map. Some articles (first reference [3] and [4, 5]) investigate Riemann-Cartan metric compatible space-time, while in second reference [3] the space-time is torsion free but with metric non-compatible connection. The third article [3] considers space-time, with nontrivial both torsion and nonmetricity. The authors assume the form of the torsion and nonmetricity and restrict considerations on D=4 space-time dimensions. The torsion is usually connected with field strength of antisymmetric field [6], while in some papers [4, 5] the trace of the torsion is related with gradient of the dilaton field.

In this article we first derive the form of the connection from the world-sheet equations of motion. It corresponds to new covariant derivative, which makes easier to perform calculations, including those in quantization procedure. We find that the string sees the space-time not as a Riemann one, but as some particular form of the affine space-time, which beside curvature also depends on torsion and nonmetricity. The features of this geometry define effective general relativity in the target space.

In Sec. 2, we formulate the theory and shortly repeat some results of ref. [7].

Starting with the known rules of the space-time parallel transport, in Sec. 3 we introduce the torsion and nonmetricity. We decompose the arbitrary connection in terms of the Christoffel one, contortion and nonmetricity. With the help of equations of motion, we find particular form of stringy torsion and stringy nonmetricity [7]. To the space-time felt by the probe string we will refer as stringy space-time.

In sec.4, we derive the form of the space-time action. We obtain the integration measure for spaces with nonmetricity from the requirements that the measure is preserved under parallel transport and that it enables integration by parts. Our integration measure is a volume-form compatible with affine connection ref.[8]. In particular application, to improve the standard measure, ref.[5] usees the torsion while we use the nonmetricity. We construct the Lagrangian linear in stringy invariants: scalar curvature, square of torsion and square of nonmetricity. We discuss the relation of the space-time action of the present paper with the space-time action of the papers [2].

Appendix A is devoted to the world-sheet geometry.

2 Canonical derivation of field equations

The closed bosonic string, propagated in arbitrary background is described by the sigma model (see [2] and [9])

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\} , \tag{2.1}$$

with x^μ-dependent background fields: metric $G_{\mu\nu}$, antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu}$ and dilaton field Φ. Here, $g_{\alpha\beta}$ is the intrinsic world-sheet metric and $R^{(2)}$ is corresponding scalar curvature. Let $x^\mu(\xi)\,(\mu = 0, 1, ..., D-1)$ be the coordinates of the D dimensional space-time M_D and $\xi^\alpha\,(\xi^0 = \tau, \xi^1 = \sigma)$ the coordinates of two dimensional world-sheet Σ, spanned by the string. The corresponding derivatives we will denote as $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$.

Let us briefly review the canonical analysis and derivation of the field equations obtained in ref. [7]. Restricting consideration to the condition $a^2 \equiv G^{\mu\nu} a_\mu a_\nu \neq 0$ $(a_\mu = \partial_\mu \Phi)$ we define the currents

$$J_{\pm\mu} = P^T{}_\mu{}^\nu j_{\pm\nu} + \frac{a_\mu}{2a^2} i^\Phi_\pm = j_{\pm\mu} - \frac{a_\mu}{a^2} j \,, \tag{2.2}$$

$$i^F_\pm = \frac{a^\mu}{a^2} j_{\pm\mu} - \frac{1}{2a^2} i^\Phi_\pm \pm 2\kappa F' \,, \qquad i^\Phi_\pm = \pi_F \pm 2\kappa \Phi' \,, \tag{2.3}$$

where

$$j_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu} x^{\nu\prime} \,, \qquad \Pi_{\pm\mu\nu} \equiv B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu} \,, \tag{2.4}$$

and

$$j = a^\mu j_{\pm\mu} - \frac{1}{2} i^\Phi_\pm = a^2 (i^F_\pm \mp 2\kappa F') \,. \tag{2.5}$$

Here π_μ and π_F are canonically conjugate momenta to the variables x^μ and F.

Up to boundary term, the canonical Hamiltonian density has the standard form

$$\mathcal{H}_c = h^- T_- + h^+ T_+ \,, \tag{2.6}$$

with the energy momentum tensor components

$$T_\pm = \mp \frac{1}{4\kappa} \left(G^{\mu\nu} J_{\pm\mu} J_{\pm\nu} + i^F_\pm i^\Phi_\pm \right) + \frac{1}{2} i^{\Phi\prime}_\pm = \mp \frac{1}{4\kappa} \left(G^{\mu\nu} j_{\pm\mu} j_{\pm\nu} - \frac{j^2}{a^2} \right) + \frac{1}{2} (i^{\Phi}_\pm{}' - F' i^\Phi_\pm) \,. \tag{2.7}$$

In spite of their complicated expressions, the same chirality energy-momentum tensor components satisfy two independent copies of Virasoro algebras,

$$\{T_\pm(\sigma), T_\pm(\bar\sigma)\} = -[T_\pm(\sigma) + T_\pm(\bar\sigma)] \delta'(\sigma - \bar\sigma) \,, \tag{2.8}$$

while the opposite chirality components commute $\{T_\pm, T_\mp\} = 0$.

2.1 Equations of motion

In ref.[7], using canonical approach, we derived the following equations of motion

$$[J^\mu] \equiv \nabla_\mp \partial_\pm x^\mu + {}^\star\Gamma^\mu_{\mp\rho\sigma} \partial_\pm x^\rho \partial_\mp x^\sigma = 0 \,, \tag{2.9}$$

$$[h^\pm] \equiv G_{\mu\nu} \partial_\pm x^\mu \partial_\pm x^\nu - 2 \nabla_\pm \partial_\pm \Phi = 0 \,, \tag{2.10}$$

$$[i^F] \equiv R^{(2)} + \frac{2}{a^2}(D_{\mp\mu}a_\nu)\partial_\pm x^\nu \partial_\mp x^\mu = 0\,, \tag{2.11}$$

where the variables in the parenthesis denote the currents corresponding to this equation. The expression

$$^\star\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\pm\nu\mu} + \frac{a^\rho}{a^2} D_{\pm\mu}a_\nu = \Gamma^\rho_{\nu\mu} \pm P^{T\rho}{}_\sigma B^\sigma_{\nu\mu} + \frac{a^\rho}{a^2} D_\mu a_\nu\,, \tag{2.12}$$

which appears in the $[J^\mu]$ equation is a generalized connection, which full geometrical interpretation we are going to investigate. Under space-time general coordinate transformations the expression $^\star\Gamma^\rho_{\pm\nu\mu}$ transforms as a connection.

The covariant derivatives with respect to the Christoffel connection $\Gamma^\rho_{\nu\mu}$ and to the connection $\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\nu\mu} \pm B^\rho_{\nu\mu}$, we respectively denote as D_μ and $D_{\pm\mu}$, while

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} = D_\mu B_{\nu\rho} + D_\nu B_{\rho\mu} + D_\rho B_{\mu\nu}\,, \tag{2.13}$$

is the field strength of the antisymmetric tensor. The projection operator which appears in eq. (2.12)

$$P^T{}_{\mu\nu} = G_{\mu\nu} - \frac{a_\mu a_\nu}{a^2} \equiv G^{D-1}_{\mu\nu}\,, \tag{2.14}$$

is the induced metric on the $D-1$ dimensional submanifold defined by the condition $\Phi(x) = const.$

In (2.9) and (2.11) we omit the currents $\pm$ indices, because $[J^\mu_+] = [J^\mu_-]$ and $[i^F_+] = [i^F_-]$ as a consequence of the symmetry relations $^\star\Gamma^\mu_{\mp\rho\sigma} = {}^\star\Gamma^\mu_{\pm\sigma\rho}$ and $D_{\mp\mu}a_\nu = D_{\pm\nu}a_\mu$.

3 The geometry of space-time seen by the probe string

In this section we introduce affine linear connection, torsion and nonmetricity (see ref. [1] for more details). With the help of string field equations we derive expressions for stringy connection, torsion and nonmetricity, recognized by the probe string.

3.1 Geometry of space-time with torsion and nonmetricity

In the curved spaces, the operations on tensors are covariant only if they are realized in the same point. In order to compare the vectors from different points we need the rule for parallel transport. The parallel transport of the vector $V^\mu(x)$, from the point x to the point $x+dx$, produce the vector $^\circ V^\mu_\parallel = V^\mu + {}^\circ\delta V^\mu$, where

$$^\circ\delta V^\mu = -{}^\circ\Gamma^\mu_{\rho\sigma} V^\rho dx^\sigma\,. \tag{3.1}$$

The variable $^\circ\Gamma^\mu_{\rho\sigma}$ is the **affine linear connection**. The covariant derivative is define in the standard form

$$^\circ DV^\mu = V^\mu(x+dx) - {}^\circ V^\mu_\parallel = dV^\mu - {}^\circ\delta V^\mu = (\partial_\nu V^\mu + {}^\circ\Gamma^\mu_{\rho\nu}V^\rho)dx^\nu \equiv {}^\circ D_\nu V^\mu dx^\nu\,. \tag{3.2}$$

The antisymmetric part of the affine connection is the **torsion**

$$^\circ T^\rho_{\mu\nu} = {}^\circ\Gamma^\rho_{\mu\nu} - {}^\circ\Gamma^\rho_{\nu\mu} \,. \tag{3.3}$$

It has a simple geometrical interpretation because it measures the non-closure of the curved "parallelogram".

The **metric tensor** $G_{\mu\nu}$ is independent variable, which enables calculation of the scalar product $VU = G_{\mu\nu}V^\mu U^\nu$, in order to measure lengths and angles.

We already learned, that covariant derivative is responsible for the comparison of the vectors from different points. What variable is responsible for comparison of the lengths of the vectors? The squares of the lengths of the vectors: $V^\mu(x)$ and its parallel transport to the point $x+dx$, $^\circ V^\mu_\|$, are defined respectively as $V^2(x) = G_{\mu\nu}(x)V^\mu(x)V^\nu(x)$ and $^\circ V^2_\|(x+dx) = G_{\mu\nu}(x+dx){}^\circ V^\mu_\| {}^\circ V^\nu_\|$. If we remember the invariance of the scalar product under the parallel transport, than the difference of the squares of the vectors is

$$^\circ\delta V^2 = {}^\circ V^2_\|(x+dx) - V^2(x) = [G_{\mu\nu}(x+dx) - G_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)]\, {}^\circ V^\mu_\| \; {}^\circ V^\nu_\| \,. \tag{3.4}$$

Up to the higher order terms we have

$$^\circ\delta V^2 = [dG_{\mu\nu}(x) - {}^\circ\delta G_{\mu\nu}(x)]V^\mu V^\nu = {}^\circ DG_{\mu\nu}V^\mu V^\nu \equiv -dx^\rho \, {}^\circ Q_{\rho\mu\nu}V^\mu V^\nu \,, \tag{3.5}$$

where we introduced the **nonmetricity** as a covariant derivative of the metric tensor

$$^\circ Q_{\mu\rho\sigma} = -{}^\circ D_\mu G_{\rho\sigma} \,. \tag{3.6}$$

Beside the length, the nonmetricity also changes the angle between the vectors V^μ_1 and V^μ_2, according to the relation

$$^\circ\delta\cos(\angle(V_1,V_2)) = \frac{-1}{2\sqrt{V_1^2V_2^2}}\left[2V_1^\rho V_2^\sigma - \left(\frac{V_1^\rho V_1^\sigma}{V_1^2} + \frac{V_2^\rho V_2^\sigma}{V_2^2}\right)(V_1V_2)\right]{}^\circ Q_{\mu\rho\sigma}dx^\mu \,. \tag{3.7}$$

Note that we performed the parallel transport of the vectors, but not of the metric tensor. It means that for the length calculation in the point $x+dx$, we used the metric tensor $G_{\mu\nu}(x+dx)$, which lives in this point, and not the tensor $G_{\mu\nu} + {}^\circ\delta G_{\mu\nu}$ obtained after parallel transport from the point x. The requirement for the equality of these two tensors is known in the literature as a metric postulate. In fact, it is just compatibility between the metric and connection, such that metric after parallel transport is equal to the local metric. Here we will not accept this requirement, because the difference of these two tensors is the origin of the nonmetricity. So, the nonmetricity measures the deformation of lengths and angles during the parallel transport.

We also define the Weyl vector as

$$^\circ q_\mu = \frac{1}{D}G^{\rho\sigma}\,{}^\circ Q_{\mu\rho\sigma} \,, \tag{3.8}$$

where D is the number of space-time dimensions. When the traceless part of the non-metricity vanishes

$$^\circ\!\not{Q}_{\mu\rho\sigma} \equiv {^\circ Q}_{\mu\rho\sigma} - G_{\rho\sigma}{^\circ q}_\mu = 0\,, \tag{3.9}$$

the parallel transport preserves the angles but not the lengths. Such geometry is known as a Weyl geometry.

Following paper [1], we can decompose the connection ${^\circ\Gamma}^\mu_{\nu\rho}$ in terms of the Christoffel connection, contortion and nonmetricity. If we introduce the Schouten braces according to the relation

$$\{\mu\rho\sigma\} = \sigma\mu\rho + \rho\sigma\mu - \mu\rho\sigma\,, \tag{3.10}$$

then the Christoffel connection, can be expressed as $\Gamma_{\mu,\rho\sigma} = \frac{1}{2}\partial_{\{\mu}G_{\rho\sigma\}}$. The contortion ${^\circ K}_{\mu\rho\sigma}$ is defined in terms of the torsion

$${^\circ K}_{\mu\rho\sigma} = \frac{1}{2}{^\circ T}_{\{\sigma\mu\rho\}} = \frac{1}{2}({^\circ T}_{\rho\sigma\mu} + {^\circ T}_{\mu\rho\sigma} - {^\circ T}_{\sigma\mu\rho})\,. \tag{3.11}$$

The Schouten braces of the nonmetricity can be solved in terms of the connection, producing

$${^\circ\Gamma}_{\mu,\rho\sigma} = \Gamma_{\mu,\rho\sigma} + {^\circ K}_{\mu\rho\sigma} + \frac{1}{2}{^\circ Q}_{\{\mu\rho\sigma\}}\,. \tag{3.12}$$

The first term is the Christoffel connection , which depends on the metric but which does not transforms as a tensor. The second one is the contortion (3.11) and the third one is Schouten braces of the nonmetricity (3.6). The last two terms transform as a tensors.

3.2 Stringy torsion and nonmetricity

The manifold M_D, together with the affine connection ${^\circ\Gamma}^\mu_{\nu\rho}$ and the metric $G_{\mu\nu}$, define the affine space-time $A_D \equiv (M_D, {^\circ\Gamma}, G)$. To the connection (2.12) we will refer as the **stringy connection** and to the corresponding space-time $S_D \equiv (M_D, {^\star\Gamma}_\pm, G)$, observed by the string propagating in the background $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, we will refer as the **stringy space-time**.

The antisymmetric part of the stringy connection is the **stringy torsion**

$${^\star T}_{\pm}{}^{\rho}_{\mu\nu} = {^\star\Gamma}_{\pm}{}^{\rho}_{\mu\nu} - {^\star\Gamma}_{\pm}{}^{\rho}_{\nu\mu} = \pm 2 P^{T\rho}{}_\sigma B^\sigma_{\mu\nu}\,. \tag{3.13}$$

It is the transverse projection of the field strength of the antisymmetric tensor field $B_{\mu\nu}$. The form of the eq.(2.13) suggests that $B_{\mu\nu}$ is a torsion potential [10].

The presence of the dilaton field Φ leads to breaking of the space-time metric postulate. The non-compatibility of the metric $G_{\mu\nu}$ with the stringy connection ${^\star\Gamma}^\mu_{\pm\nu\rho}$ is measured by the **stringy nonmetricity**

$${^\star Q}_{\pm\mu\rho\sigma} \equiv -{^\star D}_{\pm\mu}G_{\rho\sigma} = \frac{1}{a^2}D_{\pm\mu}(a_\rho a_\sigma)\,. \tag{3.14}$$

Consequently, during stringy parallel transport, the lengths and angles deformations depend on the vector field a_μ.

The stringy Weyl vector

$$
{}^\star q_\mu = \frac{1}{D} G^{\rho\sigma\star} Q_{\pm\mu\rho\sigma} = \frac{-4}{D}\partial_\mu \varphi \,, \tag{3.15}
$$

is a gradient of new scalar field φ, defined by the expression

$$
\varphi = -\frac{1}{4}\ln a^2 = -\frac{1}{4}\ln(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi)\,. \tag{3.16}
$$

The stringy angle preservation relation

$$
{}^\star \not{Q}_{\pm\mu\rho\sigma} = {}^\star Q_{\pm\mu\rho\sigma} - G_{\rho\sigma}{}^\star q_\mu = 0\,, \tag{3.17}
$$

is a condition on the dilaton field Φ. Generally, in stringy geometry both the lengths and the angles could be changed under the parallel transport.

Using the relation

$$
{}^\star K_{\pm\mu\rho\sigma} + \frac{1}{2}{}^\star Q_{\pm\{\mu\rho\sigma\}} = \pm\frac{1}{2}{}^\star T_{\mu\rho\sigma} + \frac{1}{2}{}^\star Q_{\{\mu\rho\sigma\}}\,, \tag{3.18}
$$

instead (3.12), we can write

$$
{}^\star \Gamma_{\pm\mu,\rho\sigma} = \Gamma_{\mu,\rho\sigma} \pm \frac{1}{2}{}^\star T_{\mu\rho\sigma} + \frac{1}{2}{}^\star Q_{\{\mu\rho\sigma\}}\,, \tag{3.19}
$$

where the quantities ${}^\star T_{\mu\rho\sigma} = 2P^T{}_\mu^\nu B_{\nu\rho\sigma}$ and ${}^\star Q_{\mu\rho\sigma} = -{}^\star D_\mu G_{\rho\sigma} = \frac{1}{a^2}D_\mu(a_\rho a_\sigma)$ do not depend on $\pm$ indices. In fact, the last term is ${}^\star Q_{\{\mu\rho\sigma\}} = 2\frac{a_\mu}{a^2}D_\rho a_\sigma$, so that we can recognize the starting expression (2.12).

4 The space-time action

The space-time field equations for background fields, derived as a quantum consistency condition of string theory [2], has a form

$$
\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4}B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma} + 2D_\mu a_\nu = 0\,, \tag{4.1}
$$

$$
\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2a_\rho B^\rho{}_{\mu\nu} = 0\,, \tag{4.2}
$$

$$
\beta^\Phi \equiv 4\pi\kappa\frac{D-26}{3} - R + \tfrac{1}{12}B_{\mu\rho\sigma}B^{\mu\rho\sigma} - 4D_\mu a^\mu + 4a^2 = 0\,, \tag{4.3}
$$

so that the world-sheet theory is Weyl invariant. Here $R_{\mu\nu}$, R and D_μ are space-time Ricci tensor, scalar curvature and covariant derivative, respectively, while $B_{\mu\rho\sigma}$ is field strength of the field $B_{\mu\nu}$ and $a_\mu = \partial_\mu\Phi$.

These field equations can be derived from a single space-time action

$$S = \int dx \sqrt{-G} e^{-2\Phi} [R - \frac{1}{12} B^2 + 4(\partial\Phi)^2] \,, \tag{4.4}$$

where $B^2 = B_{\mu\nu\rho} B^{\mu\nu\rho}$ and $(\partial\Phi)^2 = G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$.

The action is defined up to the total derivative. So, it depends on some constant parameter ζ

$$S_\zeta = S + \zeta \int dx \partial_\mu (\sqrt{-G} G^{\mu\nu} \partial_\nu e^{-2\Phi}) \,, \tag{4.5}$$

and can be rewritten in the form

$$S_\zeta = \int dx \sqrt{-G} e^{-2\Phi} [R - \frac{1}{12} B^2 + 4(1+\zeta)(\partial\Phi)^2 - 2\zeta D^2 \Phi] \,, \tag{4.6}$$

where $D^2 \Phi = G^{\mu\nu} D_\mu \partial_\nu \Phi$. For simplicity, in order to exclude the third term we adopt $\zeta = -1$ and obtain

$$S_{\zeta=-1} = \int dx \sqrt{-G} e^{-2\Phi} [R - \frac{1}{12} B^2 + 2 D^2 \Phi] \equiv \int dx \sqrt{-G} e^{-2\Phi} \mathcal{L} \,. \tag{4.7}$$

Using the stringy geometry introduced in the previous section, we are going to reproduce above space-time actions. Generally, it has the form

$$^\star S = \int d^D x \; ^\star\Omega \; ^\star\mathcal{L} \,, \tag{4.8}$$

where $^\star\Omega$ is a measure factor, and $^\star\mathcal{L}$ is a Lagrangian which depends on the space-time field strengths.

4.1 The space-time measure

We define the invariant measure, requiring that:

1. It is invariant under space-time general coordinate transformations.

2. It is preserved under parallel transport, which is equivalent to the condition $^\star D_{\pm\mu} {}^\star\Omega = 0$.

3. It enable integration by parts, which can be achieved with help of of the Leibniz rule and the relation

$$\int d^D x \, ^\star\Omega \, ^\star D_{\pm\mu} V^\mu = \int d^D x \partial_\mu (^\star\Omega V^\mu) \,, \tag{4.9}$$

so that we are able to use Stoke's theorem.

For Riemann and Riemann-Cartan space-times, the solution for the measure factor is well known $\Omega = \sqrt{-G}$ ($G = \det G_{\mu\nu}$). For spaces with nonmetricity, this standard measure is not preserved under the parallel transport, and requirements 2. and 3. are not satisfy. Instead to change the connection and find volume-preserving one, as has been done in ref. [1], we prefer to change the measure.

Let us try to find the stringy measure in the form ${}^\star\Omega = \Lambda(x)\sqrt{-G}$. In order to be preserved under the parallel transport with the stringy connection, it must satisfy the condition

$$ {}^\star D_{\pm\mu}(\sqrt{-G}\Lambda) = \partial_\mu(\sqrt{-G}\Lambda) - {}^\star\Gamma^\rho_{\pm\mu\rho}\sqrt{-G}\Lambda = 0\,. \tag{4.10} $$

Using the relation

$$ {}^\star\Gamma^\rho_{\pm\mu\rho} = \partial_\mu \ln\left(\sqrt{-G}e^{-2\varphi}\right) = \Gamma^\rho_{\pm\mu\rho} + \frac{D}{2}{}^\star q_\mu\,, \tag{4.11} $$

we find the equation for Λ, $\partial_\mu\Lambda = \frac{D}{2}{}^\star q_\mu\Lambda$. The fact that the stringy Weyl vector ${}^\star q_\mu$ is a gradient of the scalar field φ, defined in (3.16), help us to find the solution $\Lambda = e^{-2\varphi}$. The stringy measure factor, preserved under parallel transport with the connection ${}^\star\Gamma^\mu_{\pm\nu}$, obtains the form

$$ {}^\star\Omega = \sqrt{-G}e^{-2\varphi}\,. \tag{4.12} $$

Consequently, we have ${}^\star\Gamma^\rho_{\pm\mu\rho} = \partial_\mu \ln{}^\star\Omega$, and (4.9) is satisfied. So, if we use the stringy measure ${}^\star\Omega$ we can integrated by parts and all requirements are satisfied.

The above measure is a volume-form compatible with the connection of the ref.[8]. In our case only nonmetricity contributes to the improvement because in stringy geometry the torsion contribution vanishes, ${}^\star T_\pm{}^\rho{}_{\mu\rho} = 0$.

The measure factor in (4.7), $\sqrt{-G}e^{-2\Phi}$, has the same form as the one in the present paper and confirms the existence of some space-time nonmetricity. The requirement of the full measures equality, $\varphi = \Phi$, leads to the Liouville like equation for the dilaton field

$$ G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - e^{-4\Phi} = 0\,. \tag{4.13} $$

For $D = 2$ it turns to the real Liouville equation.

4.2 The space-time Lagrangian

We are going to reproduce the Lagrangian defined in (4.7) with suitable combinations of the stringy scalar curvature, defined in the standard way with the stringy connection (2.12)

$$ {}^\star R_\pm = R - B^2 + 2D^2\varphi - 4(\partial\varphi)^2 + e^{4\varphi}[2(aB)^2 + 2a^\mu\partial_\mu(Da) + a^\mu D_\mu(Da) + (Da)^2]\,, \tag{4.14} $$

the stringy torsion (3.13)

$$ {}^\star T^\rho{}_{\pm\mu\nu} = \pm\left[2B^\rho{}_{\mu\nu} - 2e^{4\varphi}a^\rho(aB)_{\mu\nu}\right]\,, \tag{4.15} $$

and the stringy nonmetricity (3.14)

$$^\star Q_{\pm\mu\rho\sigma} = e^{4\varphi}[D_\mu(a_\rho a_\sigma) \mp a_\rho(aB)_{\sigma\mu} \mp a_\sigma(aB)_{\rho\mu}]\,, \tag{4.16}$$

where $(aB)_{\mu\nu} = a^\rho B_{\rho\mu\nu}$, $(aB)^2 = a^\rho B_{\rho\mu\nu} a^\sigma B_\sigma{}^{\mu\nu}$ and $Da = D_\mu a^\mu$. First, we construct the corresponding invariants

$$^\star T^2_\pm \equiv {}^\star T_{\pm\mu\nu\rho}{}^\star T^{\mu\nu\rho}_\pm = 4[B^2 - e^{4\varphi}(aB)^2]\,, \tag{4.17}$$

$$^\star Q^2_\pm \equiv {}^\star Q_{\pm\mu\nu\rho}{}^\star Q^{\mu\nu\rho}_\pm = 8(\partial\varphi)^2 + 2e^{4\varphi}\left[(D_\mu a_\nu)(D^\mu a^\nu) + (aB)^2\right]\,, \tag{4.18}$$

and

$$^\star q^2 \equiv {}^\star q_\mu{}^\star q^\mu = \frac{1}{D^2} G^{\rho\sigma} Q_{\pm\mu\rho\sigma} G^{\varepsilon\eta} Q_\pm{}^\mu{}_{\varepsilon\eta} = \frac{16}{D^2}(\partial\varphi)^2\,, \tag{4.19}$$

where $^\star q_\mu$ is stringy Weyl vector defined in (3.15). Note thet all invariants are independent on $\pm$ indices and we put $^\star R_\pm = {}^\star R$, $^\star T^2_\pm = {}^\star T^2$ and, $^\star Q^2_\pm = {}^\star Q^2$.

We assume that Lagrangian is linear in these invariants and choose appropriate coefficients in front of them

$$^\star\mathcal{L} \equiv {}^\star R + \frac{1}{48}\left(11\,{}^\star T^2 - 26\,{}^\star Q^2\right) + \frac{1}{3}\left(\frac{5D}{4}\right){}^\star q^2\,, \tag{4.20}$$

in order to reproduce the expression (4.7)

$$^\star\mathcal{L} = R - \frac{1}{12}B^2 + 2D^2\varphi + \frac{1}{a^2}\left[2a^\mu\partial_\mu(Da) + a^\mu D_\mu(Da) + (Da)^2 - \frac{13}{12}(D_\mu a_\nu)(D^\mu a^\nu)\right]. \tag{4.21}$$

If the condition (4.13) is satisfied, the Lagrangian (4.21), up to the term with factor $\frac{1}{a^2}$, coincides with that defined in (4.7).

The Lagrangian (4.7) has been obtained from one-loop perturbative computations. The higher loop corrections, generally depend on the renormalization scheme (see [11]). We argue that the term proportional to $\frac{1}{a^2}$ in $^\star\mathcal{L}$ originates from the higher orders contribution. The reason is that there is difference between Lagrangian and Hamiltonian perturbative approaches, [7]. The leading order term of the Hamiltonian contains Φ dependent part proportional to $\frac{1}{a^2}$, while the leading order term of the Lagrangian is Φ independent. Because the stringy invariants of the present paper are defined from Hamiltonian form of field equations, (2.9)-(2.11), we expect that the term proportional to $\frac{1}{a^2}$ is a consequence of different perturbative approaches. Up to this term, for $\varphi = \Phi$ we have $^\star\mathcal{L} = \mathcal{L}$.

5 Conclusions

In the present paper we showed that the probe string, as an extended object, can see more space-time features then the probe particle – torsion and nonmetricity. We found their forms in terms of background fields, which defined the target space geometry recognized by the string.

The equations of motion (2.9)-(2.11) help us to obtain the explicit expression for stringy connection (2.12). It produces stringy torsion (3.13) and nonmetricity (3.14), originated from the antisymmetric field $B_{\mu\nu}$ and dilaton fields Φ, respectively.

Let us clarify how space-time geometry depends on the background fields. In the presence of the metric tensor $G_{\mu\nu}$, the space-time is of the Riemann type. Inclusion of the antisymmetric field $B_{\mu\nu}$ produces the Riemann-Cartan space-time. Appearance of the dilaton field Φ broke the compatibility between metric tensor and stringy connection. When all three background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are present, the string feels the complete stringy space-time.

Finally, we constructed the bosonic string space-time action in terms of geometrical quantities. In order to find the integration measure invariant under parallel transport we used the fact that the stringy Weyl vector is a gradient of the scalar field φ. We also derive the Lagrangian as a function of the stringy invariants: scalar curvature, torsion and nonmetricity.

We discussed the connection between our result and that of the papers [2], in spite of their different origin. The standard result is quantum and perturbative while our is classical and non perturbative. In particular, our scalar field φ, defined in (3.16), plays the role of dilaton field Φ, and has the same position in all expressions. Up to the non-linear term proportional to $\frac{1}{a^2}$, (which is a consequence of a different perturbatin theory in the Lagrangian and Hamiltonian approaches) for $\varphi = \Phi$, these two actions are equal including the dilaton factor in the integration measure.

It is well known that the dilaton dependent Weyl transformation

$$G^E_{\mu\nu} = e^{-\frac{2(\Phi_0 - \Phi)}{D-2}} G_{\mu\nu} \tag{5.22}$$

brings the Lagrangian to the Hilbert form

$$S^E = \int dx \sqrt{-G^E} \left[R - \frac{1}{12} e^{-\frac{8\Phi}{D-2}} B^2 - \frac{4}{D-2} (\partial\Phi)^2 \right]_E , \tag{5.23}$$

where index E means that all quantities are define in terms of the Einstein metric, $G^E_{\mu\nu}$. In this form of the Lagrangian, the dilaton decouples from the curvature, but it is still coupled to the torsion through the second term. As a consequence, neither of the two Lagrangians obeys the equivalence principle, so that the change from the string frame to the Einstein one, does not help us to choose a preferred definition of the metric (see second reference [9]).

Our approach prefers the so called string frame as a more fundamental, because we offered clear geometrical interpretation for it. In particular, the preservation of the integration measure under parallel transport singles out the form (4.12) for it. This is just characteristic of the string frame.

There is another reason in support of the string frame, [9]. Only when the action is written in terms of the fields originating from strings, the constant part of the dilaton, Φ_0, appears as an overall factor, as well as the coupling constant in Yang-Mills theories.

Consequently, we show that string space-time action, in terms of geometrical quantities, depends not only on curvature and torsion recognized by the probe string, but also on nonmetricity, which causes absence of the equivalence principle.

Let us mention one curiosity. It is known that the coefficient in front of the Liouville action is proportional to the central charge and measures the quantum braking of the classical symmetry. The contribution to the central charge of the anticommuting ghosts b, c corresponding to the conformal symmetry and the commuting ghosts β, γ corresponding to the superconformal symmetry are $\frac{-26}{48}$ and $\frac{11}{48}$, respectively. In definition of the Lagrangian ${}^{\star}\mathcal{L}$, (4.20), the coefficients in front of the stringy nonmetricity and the stringy torsion are just equal to the coefficients of the b, c and β, γ ghost contributions. We do not find good reason to explain this similarity, but we find interesting to mention this coincidence.

A World-sheet geometry

We used the notation of ref. [7] expressing the intrinsic world-sheet metric tensor $g_{\alpha\beta}$, in terms of the light-cone variables (h^+, h^-, F)

$$g_{\alpha\beta} = e^{2F}\hat{g}_{\alpha\beta} = \tfrac{1}{2}e^{2F}\begin{pmatrix} -2h^-h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix} . \tag{A.1}$$

The world-sheet interval has a form

$$ds^2 = g_{\alpha\beta}d\xi^\alpha d\xi^\beta = 2d\xi^+ d\xi^- \, , \tag{A.2}$$

where

$$d\xi^\pm = \frac{\pm 1}{\sqrt{2}}e^F(d\xi^1 - h^\pm d\xi^0) = e^\pm{}_\alpha d\xi^\alpha \, . \tag{A.3}$$

The quantities $e^\pm{}_\alpha$ define the light-cone one form basis, $\theta^\pm = e^\pm{}_\alpha d\xi^\alpha$, and its inverse define the tangent vector basis, $e_\pm = e_\pm{}^\alpha\partial_\alpha = \partial_\pm$.

In the tangent basis notation, the components of the arbitrary vector V_α have the form

$$V_\pm = e_\pm{}^\alpha V_\alpha = \frac{\sqrt{2}e^{-F}}{h^- - h^+}(V_0 + h^\mp V_1) \, . \tag{A.4}$$

The world-sheet covariant derivatives on tensor X_n are

$$\nabla_\pm X_n = (\partial_\pm + n\omega_\pm)X_n \, , \tag{A.5}$$

where the number n is sum of the indices, counting index $+$ with 1 and index $-$ with -1. The two dimensional covariant derivative $\nabla_\pm$ is defined with respect to the connection

$$\omega_\pm = e^{-F}(\hat{\omega}_\pm \mp \hat{\partial}_\pm F)\,, \qquad \hat{\omega}_\pm = \mp \frac{\sqrt{2}}{h^- - h^+} h^{\mp\prime}\,. \tag{A.6}$$

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